Hedging and Value-at-Risk (VaR)

Single asset VaR
Delta-VaR for portfolios
Delta-Gamma VaR
simulated VaR

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Value at Risk (VaR)

“VaR measures the **worst expected loss** over a given time interval under normal market conditions at a given confidence level.”
- Jorion (1997)

“Value at Risk is an estimate, with a given degree of confidence, of **how much one can lose** from one’s portfolio over a given time horizon.”
- Wilmott (1998)

“Value-at-Risk or VAR is a dollar measure of the **minimum loss** that would be expected over a period of time with a given probability”
- Chance(1998)

95% confidence level VaR $\Leftrightarrow$ 5% probability minimum loss
(over given horizon)

max. loss with 95% confidence $\Leftrightarrow$ min.loss with 5% probability
(for given time interval)
S.Mann, 1999
Asset price standard deviation

Assume lognormal returns: Let \( dS/S \sim \text{lognormal}(\mu, \sigma) \)
where \( \sigma \) is annualized return volatility (standard deviation)

The standard deviation of the asset price \( (S) \) over a period \( \tau \) is:
\[
S (\sigma \tau^{0.5}) = S (\sigma \sqrt{\tau})
\]

For example, let
\[
S = $ 100.00 \\
\sigma = 40\% \\
\tau = 1 \text{ week} = 1/52
\]

then the weekly standard deviation (s.d.) of the price is
\[
\text{weekly s.d.} = 100 (0.40) (0.192)^{0.5} = 40(0.139) = $ 5.55
\]

similarly,
\[
\text{daily s.d.} = 100(0.40)(1/252)^{0.5} = 40(0.063) = $ 2.52 \\
\text{monthly s.d.} = 0.40(0.40)(1/12)^{0.5} = 40(0.289) = $ 11.55
\]

S.Mann, 1999
Confidence levels and the inverse distribution function

Let VaR = \(- S (\sigma \sqrt{\tau}) N'(1 - \text{confidence level})\) 

*(for long position in underlying asset)*

where \(N'(x\%)\) = inverse cumulative distribution function for the standard normal

\(N'(x\%)\) = number of standard deviations from the mean such that the probability of obtaining a lower outcome is \(x\%\)

desired confidence level is 95%,
then \(N'(1-.95) = N'(5\%) = -1.65\)

In other words, \(N(-1.65) = 0.05\) (5%)
so \(N'(0.05) = -1.65\)

5% probability of return lower than 1.65 standard deviations below the mean

S.Mann, 1999
Confidence levels and VaR

Normal returns, $100 current price and 40% volatility:

<table>
<thead>
<tr>
<th>time period:</th>
<th>daily</th>
<th>weekly</th>
<th>monthly</th>
<th>yearly</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma S(\sqrt{\tau})$ :</td>
<td>$2.52$</td>
<td>$5.55$</td>
<td>$11.55$</td>
<td>$40.00$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C</th>
<th>1-C</th>
<th>$N(1-C)$</th>
<th>VaR</th>
<th>VaR</th>
<th>VaR</th>
<th>VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>99%</td>
<td>1%</td>
<td>-2.33</td>
<td>$5.87$</td>
<td>$12.93$</td>
<td>$26.91$</td>
<td>$96.66$</td>
</tr>
<tr>
<td>95%</td>
<td>5%</td>
<td>-1.65</td>
<td>$4.14$</td>
<td>$9.16$</td>
<td>$19.06$</td>
<td></td>
</tr>
<tr>
<td>90%</td>
<td>10%</td>
<td>-1.28</td>
<td>$3.23$</td>
<td>$7.10$</td>
<td>$14.78$</td>
<td></td>
</tr>
</tbody>
</table>

Let $\text{VaR} = -S(\sigma \sqrt{\tau}) N(1 - C)$

Ignoring drift:
assuming return is normally distributed with mean zero.
For time frames longer than a week, may need to mean-adjust.

S.Mann, 1999

**Asset price in one week**

5% probability

Weekly VaR = $9.16$
Call greeks

delta (Δ) = $MC/MS$ = call sensitivity to asset value
vba function: delta(S,K,T,r,σ)

gamma (Γ) = $MC/MS^2$ = delta sensitivity to asset value
vba function: gamma(S,K,T,r,σ)

vega = $MC/σ$ = call sensitivity to volatility
vba function: vega(S,K,T,r,σ)

theta = $MC/M_T$ = call sensitivity to time
vba function: calltheta(S,K,T,r,σ)

rho = $MC/M_r$ = call sensitivity to riskless rate
vba function: callrho(S,K,T,r,σ)

S. Mann, 1999
Put Delta and Gamma

Put-Call parity: \[ S + P = C + KB(0,T) \]

take derivative of equality with respect to asset price, \( S \).

\[
\begin{align*}
\frac{\delta S}{S} + \frac{\delta P}{P} &= \frac{\delta C}{C} + \frac{\delta [KB(0,T)]}{KB(0,T)} \\
1 + \frac{\delta P}{P} &= \frac{\delta C}{C} + 0 \\
\frac{\delta P}{P} &= \frac{\delta C}{C} - 1
\end{align*}
\]

Put Delta: \( \Delta_p = \Delta_c - 1 \)  
(put delta = call delta - 1)

Take second derivative with respect to asset price, \( S \).

\[
\begin{align*}
\frac{\delta^2 \Delta_p}{S^2} &= \frac{\delta^2 \Delta_c}{S^2} + 0 \\
\frac{\delta^2 P}{P^2} &= \frac{\delta^2 C}{C^2} + 0
\end{align*}
\]

Put Gamma: \( \Gamma_p = \Gamma_c \)  
(put gamma = call gamma)
**Put vega and theta**

Put-Call parity: \( S + P = C + KB(0,T) \)

take derivative of equality with respect to volatility, \( \sigma \).

\[
\begin{align*}
\frac{\partial S}{\partial \sigma} + \frac{\partial P}{\partial \sigma} &= \frac{\partial C}{\partial \sigma} + \frac{\partial [KB(0,T)]}{\partial \sigma} \\
0 + \frac{\partial P}{\partial \sigma} &= \frac{\partial C}{\partial \sigma} + 0 \\
\frac{\partial P}{\partial \sigma} &= \frac{\partial C}{\partial \sigma}
\end{align*}
\]

**Put Vega = Call Vega**

take derivative of equality with respect to time, \( T \).

\[
\begin{align*}
\frac{\partial S}{\partial T} + \frac{\partial P}{\partial T} &= \frac{\partial C}{\partial T} + \frac{\partial [KB(0,T)]}{\partial T} \\
0 + \frac{\partial P}{\partial T} &= \frac{\partial C}{\partial T} + K \frac{\partial [KB(0,T)]}{\partial T} \\
\frac{\partial P}{\partial T} &= \frac{\partial C}{\partial T} + K \frac{e^{-rT}}{T}
\end{align*}
\]

**Put theta = Call theta + \( rK e^{-rT} \)**

Call theta is always negative; put theta is ambiguous

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Put greeks

\[
\text{delta (}\Delta\text{)} = \frac{\Delta P}{\Delta S} = \text{put sensitivity to asset value} \\
\text{put delta} = \text{call delta} - 1 \\
\Delta P = \Delta c - 1 \\
\text{function: } \delta(S,K,T,r,\sigma) - 1
\]

\[
\text{gamma (}\Gamma\text{)} = \frac{\Gamma P}{\Gamma S^2} = \text{delta sensitivity to asset value} \\
\text{put gamma} = \text{call gamma} \\
\text{function: } \gamma(S,K,T,r,\sigma)
\]

\[
\text{vega} = \frac{\Delta P}{\Delta \sigma} = \text{put sensitivity to volatility} \\
\text{put vega} = \text{call vega} \\
\text{function: } \nu(S,K,T,r,\sigma)
\]

\[
\text{theta} = \frac{\Delta P}{\Delta T} = \text{put sensitivity to time} \\
\text{vba function: } \psi(S,K,T,r,\sigma)
\]

\[
\text{rho} = \frac{\Delta P}{\Delta r} = \text{put sensitivity to riskless rate} \\
\text{vba function: } \rho(S,K,T,r,\sigma)
\]
**Call VaR: Delta-VaR**

VaR for long position in underlying asset:

\[ \text{VAR} = -\sigma S(\sqrt{\tau}) N'(1-C) \]

Since \( \Delta \) represents call exposure to underlying,  
*(for small moves, if asset increases $1, call increases $\Delta$)*

define \[ \text{Delta-VaR} = -\sigma \Delta S(\sqrt{\tau}) N'(1-C), \]
e.g. \[ \text{Delta-VaR} = \Delta \text{VaR} \]

Example: $100 stock, 40% volatility, 95% daily VAR = $4.14
Examine 95% daily \( \Delta \)-VaR for the following 110-day calls:

<table>
<thead>
<tr>
<th>Strike</th>
<th>Value</th>
<th>( \Delta )</th>
<th>95% ( \Delta )-VaR</th>
<th>(( \Delta )-VaR)/Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>11.96</td>
<td>0.66</td>
<td>2.72</td>
<td>22.8%</td>
</tr>
<tr>
<td>100</td>
<td>9.36</td>
<td>0.57</td>
<td>2.35</td>
<td>25.1%</td>
</tr>
<tr>
<td>105</td>
<td>7.21</td>
<td>0.48</td>
<td>1.99</td>
<td>27.6%</td>
</tr>
<tr>
<td>110</td>
<td>5.47</td>
<td>0.40</td>
<td>1.64</td>
<td>30.0%</td>
</tr>
</tbody>
</table>

S. Mann, 1999
Delta-VaR for portfolios including options

VaR for long position in underlying asset:

\[ \text{VaR} = -S \left( \sigma \sqrt{\tau} \right) N \left( 1-C \right) \]

and define \( \Delta \text{-VaR} = -\Delta S \left( \sigma \sqrt{\tau} \right) N \left( 1-C \right) \),

then define \( \text{Portfolio } \Delta \text{-VaR} = -\sum n_i \Delta_i S \left( \sigma \sqrt{\tau} \right) N \left( 1-C \right) \)

for positions \( n_i \), \( \{i = 1\ldots N\} \), all written on the same underlying.

This definition holds for puts, calls, and units of underlying.

example:

$100 stock, 40\%$ volatility, \textbf{95\% daily VaR} = $4.14$

Examine \( \Delta \text{-VaR} \) for straddle - long put \( (n_p = 1) \) and long call \( (n_c = 1) \):

<table>
<thead>
<tr>
<th>Strike</th>
<th>Value</th>
<th>( \Delta )</th>
<th>95% ( \Delta \text{-VaR} )</th>
<th>((\Delta \text{-VaR})/\text{Value})</th>
</tr>
</thead>
<tbody>
<tr>
<td>call</td>
<td>100</td>
<td>9.36</td>
<td>0.57</td>
<td>2.35</td>
</tr>
<tr>
<td>put</td>
<td>100</td>
<td>8.04</td>
<td>-0.43</td>
<td>-1.79</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>17.40</td>
<td>0.14</td>
<td>0.56</td>
</tr>
</tbody>
</table>

S.Mann, 1999
Short Call position Delta-hedged

Portfolio $\Delta$-VaR = $(-\sum n_i \Delta_i) S(\sigma \sqrt{\tau}) N'(1-C)$

sell call: 115 strike, price = $14.00, implied volatility = 49%
hedge: buy $\Delta$ shares, priced at $116.625$

95% weekly $\Delta$VaR:
   a) compute $S(\sigma \sqrt{\tau}) N'(5\%) = $13.04/share (~11%)
   b) find position delta ($\sum n\Delta$)

<table>
<thead>
<tr>
<th>Position</th>
<th>n</th>
<th>cost</th>
<th>$\Delta$</th>
<th>n$\Delta$</th>
<th>$\Delta$VaR</th>
</tr>
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<tbody>
<tr>
<td>short call</td>
<td>-100</td>
<td>$-1,400$</td>
<td>0.593</td>
<td>-59.32</td>
<td>$-773.18$</td>
</tr>
<tr>
<td>long shares</td>
<td>60</td>
<td>6,998</td>
<td>1.000</td>
<td>60.0</td>
<td>782.10</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>$5,609$</td>
<td></td>
<td>0.68</td>
<td>$8.92$</td>
</tr>
</tbody>
</table>

95% weekly $\Delta$VaR/cost = $9/5609 = 0.16$

S.Mann, 1999
## Short Call Delta-hedged

<table>
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<tr>
<th>Position</th>
<th>$n$</th>
<th>cost</th>
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<td></td>
<td></td>
<td>0.68</td>
<td>8.92</td>
</tr>
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Stock 95% weekly VaR = $13.04.

What if stock rises $1.65 \sigma \sqrt{\tau} = $13.04 to new price of 129.67?

If no time elapses, the position changes to:

<table>
<thead>
<tr>
<th>Position</th>
<th>$n$</th>
<th>value</th>
<th>$\Delta$</th>
<th>n$\Delta$</th>
<th>$\Delta$VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>short call</td>
<td>-100</td>
<td>-2,260</td>
<td>0.736</td>
<td>-73.60</td>
<td>-1067.00</td>
</tr>
<tr>
<td>long shares</td>
<td>60</td>
<td>7,780</td>
<td>1.000</td>
<td>60.0</td>
<td>870.00</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>5,520</td>
<td>-13.60</td>
<td>197.00</td>
<td></td>
</tr>
</tbody>
</table>

**loss in value = $89.00 !**

S.Mann, 1999
Adjusting Δ-VaR for nonlinearity

Use Taylor-series expansion of function: given value at $x_0$, use derivatives of function to approximate value of function at $x$:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + (1/2)f''(x_0)(x-x_0)^2 + ...$$

Incorporate curvature via
Second-order Taylor series expansion of option price around the stock price (quadratic approximation):

$$C(S + dS) = C(S) + (\frac{\partial C}{\partial S})dS + (0.5) \frac{\partial^2 C}{\partial S^2} (dS)^2$$

so that

$$C(S + dS) - C(S) = \Delta dS + 0.5 \Gamma (dS)^2 + ...$$

or

$$dC = \Delta dS + 0.5 \Gamma (dS)^2 + ...$$
we want an expression for the standard deviation of an option’s price

Quadratic approximation of change in call value for given change in stock value:

\[ dC = \Delta dS + 0.5 \Gamma (dS)^2 + \ldots \]

then the variance of the call’s price, \( V(dC) \) for given time, \( \tau \), is:

\[ V(dC) = \Delta^2 V(dS) + 0.5 \left[ \Gamma V(dS) \right]^2 \]

note that \( V(dS) = S^2 \sigma^2 \tau = \) variance of stock price over period \( \tau \)

so that \( V(dC) = \Delta^2 S^2 \sigma^2 \tau + 0.5 \left[ \Gamma S^2 \sigma^2 \tau \right]^2 \)

thus the standard deviation of the option price is:

\[ \text{s.d. (}dC\text{)} = \left\{ \Delta^2 S^2 \sigma^2 \tau + 0.5 \left[ \Gamma S^2 \sigma^2 \tau \right]^2 \right\}^{(1/2)} \]

(what if \( \Gamma = 0? \)) if \( \Gamma = 0 \) then \( \text{s.d. (}dC\text{)} = \Delta S \sigma \sqrt{\tau} \)

S. Mann, 1999
Delta-Gamma VaR

Taylor expansion leads to quadratic approximation for the standard deviation of the option price:

\[
\text{s.d.}(dC) = \left\{ \Delta^2 S^2 \sigma^2 \tau + 0.5 \left[ \Gamma S^2 \sigma^2 \tau \right]^2 \right\}^{(1/2)}
\]

Note that if \( \Gamma = 0 \) then s.d.(\( dC \)) = \( \Delta S \sigma \sqrt{\tau} \)

Examine Delta-VaR: \( \Delta-\text{VaR} = \Delta S \sigma \sqrt{\tau} \ N'(1-C) \)

delta-VaR is a linear operator on underlying asset VaR

Delta-Gamma VaR = option s.d. \( N'(1-C) \)

\[
= \left[ \Delta^2 S^2 \sigma^2 \tau + 0.5 \left[ \Gamma S^2 \sigma^2 \tau \right]^2 \right]^{(1/2)} \ N'(1-C)
\]

delta-Gamma VaR is nonlinear function of asset VaR
Delta-Gamma VaR = option s.d. \( N'(1-C) \)

\[
= [\Delta^2 S^2 \sigma^2 \tau + 0.5 (\Gamma S^2 \sigma^2 \tau)^2]^{(1/2)} N'(1-C)
\]

Since Delta-Gamma VaR is nonlinear function of Delta and Gamma, cannot simply add VaR, as in Delta-VaR.

For portfolio of linear and/or nonlinear derivatives on a single underlying:

Solution: 1) compute portfolio Delta \( \Delta_p \)

2) compute portfolio Gamma \( \Gamma_p \)

3) compute the quadratic approximation to the portfolio standard deviation

\[
[\Delta_p^2 S^2 \sigma^2 \tau + 0.5 (\Gamma_p S^2 \sigma^2 \tau)^2]^{(1/2)}
\]

4) multiply the portfolio standard deviation by \( N'(1-C) \) to get \( \Delta-\Gamma \) VaR

S.Mann, 1999
Delta-Gamma VaR for delta-hedged short call

<table>
<thead>
<tr>
<th>Position</th>
<th>n</th>
<th>cost</th>
<th>Δ</th>
<th>nΔ</th>
<th>Γ</th>
<th>nΓ</th>
</tr>
</thead>
<tbody>
<tr>
<td>short call</td>
<td>-100</td>
<td>$-1,400</td>
<td>0.593</td>
<td>-59.32</td>
<td>0.012</td>
<td>-1.2</td>
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<td>long shares</td>
<td>60</td>
<td>6,998</td>
<td>1.0</td>
<td>60.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>0.68</td>
<td></td>
<td>-1.2</td>
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</table>

Find quadratic approximation to position standard deviation:

\[
\frac{1}{2} \left[ \Delta^2 \sigma^2 \tau + 0.5 \left( \Gamma \sigma^2 \tau \right)^2 \right]^{(1/2)} = \left[ (0.68)^2 (62.80) + 0.5 (-1.2 \times 62.80)^2 \right]^{(1/2)}
\]

\[
= \left[ 0.4679 (62.80) + 0.5 (-78.03)^2 \right]^{(1/2)}
\]

\[
= \left[ 29.3875 + 0.5 (6088.81) \right]^{(1/2)}
\]

\[
= \left[ 29.3875 + 3044.40 \right]^{(1/2)}
\]

\[
= 55.44
\]

(using \( S = 116.625 \), \( \sigma = 49\% \), \( \tau = 1 \) week, so that \( S\sigma\sqrt{\tau} = \$7.92 \); \( S^2\sigma^2\tau = \$62.80 \))

Multiply the portfolio standard deviation by \( N'(1-C) = -1.65 \) for \( C=95\% \)

Delta-Gamma VaR = 55.44 x (-1.65) = \(-91.19\)
## Delta-Gamma VaR compared to Delta VaR

<table>
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<tr>
<th>Position</th>
<th>n</th>
<th>cost</th>
<th>Δ</th>
<th>nΔ</th>
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Stock 95% weekly VaR = $13.04.

**What if stock rises 1.65 (Sσ√τ) = $13.04,** to new price of 129.67?

the position changes to: (assume no time lapse)

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**loss in value = $89.00** - a bit less than Δ–Γ VaR = 91.19

If position is unchanged, Γ moves to -0.9 and the new Δ–Γ VaR = $214.90

S.Mann, 1999
Monte Carlo
simulate distributions of underlying assets; for each simulated outcome use pricing model to evaluate portfolio (underlying and options).

Heavy computational requirements.

Requires many inputs (e.g. variance, correlations)
Assumes specific return-generating process (e.g. normal).

Should use expected returns
(not risk-neutral drift, as in monte carlo pricing).
Alternative VaR approaches - Bootstrapping

Bootstrapping (historical)

database of return vectors (e.g., $r_t = r_{1t}, r_{2t}, \ldots, r_{Nt}$)

randomly sample from historical returns to generate return sequences
- potential future scenarios based on historical data.

All asset returns on given date are kept together - thus the bootstrap captures historical correlations between assets.

Incorporates correlation, but not autocorrelation.
Allows for non-normality.

Data requirements are large.

S. Mann, 1999
references


